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Homework 3

Due February 1st on paper at the beginning of class. Justify your answers. Please let me know if you have a question or find a mistake. There are some hints on the second page.

1. Borthwick Exercise 4.1.
2. Give an alternate proof of Borthwick Theorem 4.7 by the following steps.
   (a) Solve $w_t(t,x) - cw_x(t,x) = f(t,x)$, $w(0,x) = 0$ by the method of characteristics.
   (b) Solve $u_t(t,x) + cu_x(t,x) = w(t,x)$, $u(0,x) = 0$ by the method of characteristics and plug in the answer from part (a).
3. Show that, if $r = |x|$ with $x \in \mathbb{R}^n$, and $\Delta = \partial_{x_1}^2 + \cdots \partial_{x_n}^2$, then

$$\Delta f(r) = f''(r) + \frac{n-1}{r} f'(r).$$

4.

Prove that, among all possible dimensions, only in three dimensions can one have distortionless spherical wave propagation with attenuation. This means the following. A spherical wave in n-dimensional space satisfies the PDE

$$u_{tt} = c^2 \left(u_{rr} + \frac{n-1}{r} u_r \right),$$

where r is the spherical coordinate. Consider such a wave that has the special form $u(r, t) = \alpha(r) f(t - \beta(r))$, where $\alpha(r)$ is called the attenuation and $\beta(r)$ the delay. The question is whether such solutions exist for "arbitrary" functions f.

(a) Plug the special form into the PDE to get an ODE for f.
(b) Set the coefficients of f'', f', and f equal to zero.
(c) Solve the ODEs to see that $n = 1$ or $n = 3$ (unless $u \equiv 0$).
(d) If $n = 1$, show that $\alpha(r)$ is a constant (so that "there is no attenuation").

(T. Morley, *American Mathematical Monthly*, Vol. 27, pp. 69–71, 1985)

Hints:

4.1. Use the form (4.14) of the solution, and use the approach below from Strauss' book. (But you must find a version of the identity adapted to Borthwick's notation and constants.)

If $u(x,t)$ satisfies the wave equation $u_{tt} = u_{xx}$, prove the identity

$$u(x+h,\ t+k) + u(x-h,\ t-k) = u(x+k,\ t+h) + u(x-k,\ t-h)$$

for all x, t, h, and k. Sketch the quadrilateral Q whose vertices are the arguments in the identity.

2b. Switch the order of integration and substitute.

4. For 4c, start with the equation coming from f, then do the one for f'', then plug into the one for f' afterwards. To solve $y''(r) + ar^{-1}y'(r) = 0$, recall that it is enough test $y(r) = r^m$ and solve for m and use the fact that the space of solutions has dimension two; if you don't get two different values of m use r^m and $r^m \log r$. Incidentally this is also from Strauss' book.